Critical Coupling in (1+1)-Dimensional Light-Front ϕ^4 Theory. II -effects of non-diagonal interactions-

Kazuto Oshima E-mail address: oshima@nat.gunma-ct.ac.jp@
Gunma National College of Technology, Maebashi 371-8530, JAPAN

Abstract

Spontaneous symmetry breaking in (1+1)-dimensional ϕ^4 theory is studied with discretized light-front quantization. Taking effects of non-diagonal interactions into account, the first few terms of the commutation relations $[a_0, a_n]$ are recalculated in the $\hbar$ expansion. Our result of the critical coupling is still consistent with the equal-time result $22\mu^2/\hbar \leq \lambda_{\rm cr} \leq 55.5\mu^2/\hbar$. We also have examined effects of regarding the ratio of the bare coupling constant to a renormalized mass as an independent parameter in the $\hbar$ expansion.

1 Introduction

Light-front QCD appears to be a hope of understanding hadrons from a field-theoretical point of view.[1)] Light-front field theory(LFT) is expected to be equivalent to the ordinary field theory not only in the perturbative region,[2)] but also in the non-perturbative region. Spontaneous symmetry breaking(SSB) is a typical phenomenon in the non-perturbative region. One of the most important feature of LFT is that the true vacuum is identical to the Fock vacuum. Therefore it appears that SSB does not occur in LFT, but it is now widely believed that a rich low-energy structure of the theory can be explained through zero modes of field operators. In LFT, zero modes of some field operators are not independent variables,[3)] but obey certain constraint equations and are given by complex functions of non-zero modes. In the ordinary equal-time formulation, (1+1)-dimensional ϕ^4 model exhibits spontaneous breakdown of Z_2 symmetry.[4)] It is therefore an important issue to confirm that the phase transition of ϕ^4 model is treated correctly in the light-front formulation. Several authors[5)−8)] have investigated the phase transition of the ϕ^4 model in the light-front formulation. In this formulation a non-zero vacuum expectation value of the zero mode is a signal of the phase transition. In the previous paper[9)] the author and Yahiro have pointed out that if we can calculate the commutation relations of the zero mode and the non-zero modes $[a_0, a_n]$, we can compute the critical coupling of the $(1+1)$-dimensional light-front ϕ^4 model. In Ref. 9) we calculated the main part of $[a_0, a_n]$ to $O(\hbar^4)$ in the $\hbar$ expansion. We neglected non-diagonal inter-

actions, which cause transitions among different modes, since they only give vanishing contributions for certain vacuum expectation values at the first level of the interactions. In this paper we take the non-diagonal interactions into account besides diagonal interactions, since the non-diagonal interactions give non-vanishing contributions at the second level of the interactions and subsequently the critical coupling will shift. Taking all the interactions into account we evaluate the critical coupling within the third-order corrections of the $\hbar$ expansion. We also calculate a renormalized mass using front-form pertubation theory and compute the critical coupling in terms of the renormalized mass.

This paper is organized as follows: In § 2 we denote some conventions of light-front theory and formulate $(1+1)$-dimensional ϕ^4 model. In § 3 we calculate the first few terms of the commutation relations $[a_0, a_n]$ in the $\hbar$ expansion. In § 4 we calculate the vacuum expectation value of the zero mode and compute the critical coupling. In § 5 we calculate the relation between a bare mass and a renormalized mass and evaluate the critical coupling in terms of the renormalized mass. In the last section we give summary. The text is supplemented by five appendices. In these appendices, several relevant commutation relations are given to calculate $[a_0, a_n]$. These commutation relations are related each other.

2 The model and formulation

We define the light-front coordinates $x^{\pm} = (x^0 \pm x^1)/\sqrt{2}$. The Lagrangian density of $(1+1)$-dimensional ϕ^4 theory is written

$$\mathcal{L} = \partial_+ \phi \partial_- \phi - \frac{\mu^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4. \tag{1}$$

We put the quantum system in a box of length d and impose periodic boundary conditions.[3),10)] The field operator ϕ is then expanded as

$$\phi(x) = \frac{1}{\sqrt{4\pi}} a_0 + \sum_{n \neq 0} \frac{1}{\sqrt{4\pi |n|}} a_n(x^+) e^{i k_n^+ x^-}, \tag{2}$$

where $k_n^+ = 2\pi n/d$. The coefficients of the expansion are operators which satisfy the canonical commutation relations,[3),7),8)]

$$[a_k, a_l] = [a_k^\dagger, a_l^\dagger] = 0, \qquad [a_k, a_l^\dagger] = \hbar \delta_{k,l}, \qquad k, l > 0, \tag{3}$$

where $a_k^\dagger = a_{-k}$. The zero-mode operator a_0 is not an independent quantity, since it governs the constraint equation[3),7)]

$$\begin{aligned} 0 \quad = \quad & a_0^3 + g a_0 + \sum_{n \geq 1} \frac{1}{n} (a_0 a_n a_n^\dagger + a_0 a_n^\dagger a_n + a_n a_n^\dagger a_0 + a_n^\dagger a_n a_0 \\ & + a_n a_0 a_n^\dagger + a_n^\dagger a_0 a_n - 3\hbar a_0) + 6 \textstyle\sum_3 \equiv \Phi(a, a^\dagger), \end{aligned} \tag{4}$$

where $g \equiv 24\pi\mu^2/\lambda$ is the only parameter of the theory and we have added the term $\sum_{n\geq 1} \frac{-3}{n}\hbar a_0$ to remove tadpole divergences and $\sum_n$ are defined as

$$\textstyle\sum_n \equiv \frac{1}{n!} \sum_{k_1, k_2 \cdots, k_n \neq 0} \frac{\delta_{k_1+k_2+\cdots+k_n,0}}{\sqrt{|k_1 k_2 \cdots k_n|}} : a_{k_1} a_{k_2} \cdots a_{k_n} : . \tag{5}$$

The commutation relations (3) and the constraint equation (4) were obtained by quantizing the classical system with the Dirac-Bergmann quantization procedure. This procedure does not specify an operator ordering in the constraint equation (4). Hence we have assumed the Weyl (symmetric) ordering.[11)]

In the previous paper[9)] the author and Yahiro have emphasized the importance of the commutation relations $[a_0, a_n]$ in calculating the vacuum expectation value $\langle 0|a_0|0\rangle$. We obtained the first few terms of $[a_0, a_n]$ in the $\hbar$ expansion, neglecting the cubic term $\sum_3$ in (4) which causes the non-diagonal interactions for simplicity. In fact, at the first level of the interactions, it does not contribute to the vacuum expectation value $\langle 0|\Phi(a, a^\dagger)|0\rangle$. We also used $[a_0, a_m^\dagger a_m] = 0$, which hold when the cubic term is absent. In this paper we recalculate $\langle 0|\Phi(a, a^\dagger)|0\rangle$ within $O(\hbar^5)$ without neglecting the cubic term. The cubic term gives non-zero contributions at the second level of the interactions.

3 Commutation Relations $[a_0, a_n]$

In this section we will calculate $[a_0, a_n]$ from

$$[\Phi(a, a^\dagger), a_n] = 0. \tag{6}$$

Using the commutation relations (3), Eq.(6) is written explicitly as

$$\begin{aligned} & A[a_0, a_n] | \frac{1}{n} a_0 a_n \hbar + \frac{1}{n}[a_0, a_n]\hbar | \frac{1}{6} \sum_{m\geq 1} \frac{1}{m} [[[a_0, a_n], a_m], a_m^\dagger] \\ & | \frac{1}{2} \sum_{k\neq 0,n} \frac{1}{\sqrt{|nk(n-k)|}} a_k a_{n-k} \hbar @ + \frac{1}{2} \sum_{m\geq 1} \frac{1}{m} [[a_0, a_m^\dagger a_m], a_n] \\ & + \frac{1}{2} a_0 [[a_0, a_n], a_0] + \frac{1}{6} [[[a_0, a_n], a_0], a_0] = 0, \end{aligned} \tag{7}$$

where

$$A = \frac{g}{6} + \frac{1}{2} a_0^2 + \sum_{m\geq 1} \frac{1}{m} a_m^\dagger a_m .^{1} \tag{8}$$

In this paper, we will use the letter m as a natural number and the letters k and l as integers. We mainly use the letter n as a natural number except for some cases, where we use the letter n as a non-zero integer. The vacuum expectation

[1] The operator A in Ref. 9) is different from A (1) by a factor 6.

value of the fifth term($\sim \sum a_k a_{n-k}$) in (7) which originates from the cubic term $\sum_3$ in (4) vanishes and the last three terms in (7) vanish or become higher order in a_0 as long as the fifth term is neglected(a_0 is very small around the critical point). In this reason in the previous paper[9)] we have neglected the last four terms in (7).

We will fix $[a_0, a_n]$ order by order in the $\hbar$ expansion. At the lowest level we have

$$[a_0, a_n] = \frac{1}{nA} a_0 a_n \hbar + \frac{1}{2A} \sum_{k \neq 0,n} \frac{1}{\sqrt{|kn(n-k)|}} a_k a_{n-k} \hbar + O(\hbar^2). \tag{9}$$

The right hand side of the commutation relations $[a_0, a_n]$ can be divided into two groups by the powers of the zero mode a_0. We call such terms as the first term in (9) diagonal that contain odd powers of a_0. In the contrast, we call such terms as the second term in (9) non-diagonal that contain even powers of a_0. Note that the powers of a_0 can increase only through the commutation relations $[\frac{1}{A}, a_n]$.

Since there is no *a priori* way of fixing the operator ordering in (9), we have fixed the operator ordering so that we can calculate the vacuum expectation value easily. Although the difference of the operator ordering affects the higher-order terms in the expansion, the vacuum expectation values $\langle 0|[a_0, a_0]a_n^\dagger|0\rangle$ would not be affected. We get $[a_0, a_n^\dagger]$ by Hermite conjugation of (9)

$$\begin{aligned} [a_0, a_n^\dagger] &= -a_n^\dagger a_0 \frac{1}{nA}\hbar - \sum_{k \neq 0,n} \frac{1}{\sqrt{|kn(n-k)|}} a_{n-k}^\dagger a_k^\dagger \frac{1}{2A}\hbar + O(\hbar^2) \\ &= -\frac{1}{nA} a_0 a_n^\dagger \hbar - \frac{1}{2A} \sum_{k \neq 0,n} \frac{1}{\sqrt{|kn(n-k)|}} a_{-k} a_{k-n} \hbar + O(\hbar^2).@ \end{aligned} \tag{10}$$

Owing to the difference of the operator ordering, the $O(\hbar^2)$ term in the second line differs from that in the first line. We adopt the operator ordering in the second line, where A and a_0 are rearranged in order from the left. By noting $a_n^\dagger = a_{-n}$ and changing some variables as $k \to -k$, (9) and (10) can be combined as

$$[a_0, a_n] = \frac{1}{A}\frac{1}{n} a_0 a_n \hbar + \frac{1}{2A} \sum_{k \neq 0,n} \frac{\epsilon(n)}{\sqrt{|kn(n-k)|}} a_k a_{n-k} \hbar + O(\hbar^2), n \neq 0, \tag{11}$$

where $\epsilon(n) = 1 (n > 0), \epsilon(n) = -1 (n < 0)$. Note that the second term in (11) are quadratic in the non-zero modes due to the non-diagonal interactions.

Before proceeding to higher order in $\hbar$, let us examine what terms should be left in our approximation. We consider the following expansion (see Ref. 9))

$$[a_0, a_n] \equiv \sum_{p \geq 1} [a_0, a_n]_p \hbar^p$$

$$
\begin{aligned}
&= \sum_{p\geq 1}\frac{\alpha_p(n)}{A^p}a_0a_n\hbar^p + \sum_{p\geq 3}\sum_{l\geq 1}\frac{\beta_{p,l}(n)}{A^{p+1}}a_l^\dagger a_la_0a_n\hbar^p \\
&+ \sum_{p\geq 5}\sum_{k,l\geq 1}\frac{\gamma_{p,kl}(n)}{A^{p+2}}a_k^\dagger a_ka_l^\dagger a_la_0a_n\hbar^p + \frac{1}{2A}\sum_{k\neq 0,n}\frac{1}{\sqrt{|kn(n-k)|}}a_ka_{n-k}\hbar \\
&+ (\mathrm{higher-order\ terms}) + O(a_0^3), \quad n\geq 1, \qquad (12)
\end{aligned}
$$

where $\alpha_p(n), \beta_{p,l}(n)$ and $\gamma_{p,kl}$ are the coefficients of the diagonal terms to be determined. Now we comment on the the expansion (12). Since $[\frac{1}{A}, a_l]\ni \frac{1}{A^2}a_l\hbar$, $[a_0, a_l]\ni \frac{1}{A}a_0a_l\hbar$ and the same relations hold for $a_l^\dagger$, the terms $\frac{1}{A^4}a_l^\dagger a_la_0a_n\hbar^3$ in (12) are produced from the fourth term in (7). The non-diagonal terms do not directly give non-zero contributions, but they affect the diagonal terms at higher orders, and at O($\hbar^2$) we will write down $[a_0, a_n]$ completely. At O($\hbar^3$) and O($\hbar^4$) only the terms $\frac{\alpha_p(n)}{A^p}a_0a_n\hbar^p (p = 3, 4)$ give non-zero contributions to the vacuum expectation value $\langle 0|\Phi(a, a^\dagger)|0\rangle$ within O($\hbar^5$). For example, the terms $\frac{\beta_{3,l}(n)}{A^4}a_l^\dagger a_la_0a_n\hbar^3$ can indirectly give non-zero contributions to $\langle 0|\Phi(a, a^\dagger)|0\rangle$ only at $O(\hbar^6)$. At O($\hbar^3$) and O($\hbar^4$) therefore we only pay attention to terms that reduce to $a_0a_n\hbar^3$ or $a_0a_n\hbar^4$.

At $O(\hbar^2)$ only four terms in the constraint equation (7) are concerned. These terms are calculated from $[a_0, a_n]_1$ and $[\frac{1}{A}, a_n]_1$. The fundamental commutation relations $[\frac{1}{A}, a_n]_1$ and some secondary commutation relations are denoted in Appendices A, B and C. From (11), (51) and (61) we obtain

$$
\begin{aligned}
& [a_0, a_n]_2 \\
&= -\frac{1}{n^2A^2}a_0a_n + \frac{1}{2n^2A^3}a_0^3a_n \\
&\quad +\frac{\epsilon(n)}{8A^3}\sum_{k\neq 0,n}\sum_{l\neq 0,n-k}\frac{a_0(a_la_{n-k-l}a_k + a_ka_la_{n-k-l})}{(n-k)\sqrt{|nkl(n-k-l)|}} \\
&- \frac{\epsilon(n)}{4}\sum_{k\neq 0,n}\Big(\frac{1}{n}+\frac{1}{k}+\frac{1}{n-k}\Big)\Big(\frac{1}{A^2}-\frac{1}{A^3}a_0^2\Big)\frac{1}{\sqrt{|nk(n-k)|}}a_ka_{n-k} \\
&- \sum_{m=1}\frac{\epsilon(m)}{4mn}\Big(\frac{1}{A^3}-\frac{1}{A^4}a_0^2\Big)\sum_{k\neq 0,m}\frac{a_n(a_{-m}a_ka_{m-k} - a_ma_{-k}a_{-m+k})}{\sqrt{|mk(m-k)|}} \\
&+ \frac{\epsilon(n)}{8A^4}\sum_{m=1}\sum_{l\neq 0,n}\sum_{k\neq 0,m}\frac{\epsilon(m)}{m\sqrt{|ln(n-l)km(m-k)|}} \\
&\times\ a_0[a_{n-l}a_l, a_{-m}a_ka_{m-k} - a_ma_{-k}a_{k-m}], \quad n\neq 0. \qquad (13)
\end{aligned}
$$

Although the last term in (13) belongs to $[a_0, a_n]_3$, we write it down here so that we may forget it at $O(\hbar^3)$. We have left the symbols $\epsilon(m)(= 1)$ as they are, since they are useful to distinguish the non-diagonal terms from the diagonal terms.

At $O(\hbar^3)$ six terms in the constraint equation (7) are concerned. To obtain $[a_0, a_n]_3$ several commutation relations at $O(\hbar)$ and $O(\hbar^2)$ are needed. Relevant commutation relations are denoted in Appendices. Non-diagonal terms in $[a_0, a_n]_3$ do not give non-zero contributions to the vacuum expectation value $\langle 0|\Phi(a, a^\dagger)|0\rangle$ within $O(\hbar^5)$. Neglecting irrelevant terms, from (13),(52),(61),(62) and (67) we have

$$
\begin{aligned}
&[a_0, a_n]_3 \\
&= \Big(\frac{4}{3n^3} + \frac{\zeta(2)}{3n}\Big)\frac{1}{A^3} a_0 a_n \\
&-\frac{1}{A^4}\sum_{k\neq 0,n}\sum_{l\neq 0,n-k}\Big(\frac{1}{6n} + \frac{11}{48k} + \frac{5}{48l} + \frac{1}{8(n-k)} + \frac{5}{48(n-k-l)}\Big) \\
&\times \frac{1}{(n-k)\sqrt{|nkl(n-k-l)|}} a_0 (a_l a_{n-k-l} a_k + a_k a_l a_{n-k-l}) \\
&+\frac{1}{8A^4}\sum_{m=1}\sum_{k\neq 0,n,m,m-n}\frac{1}{m(m-k)\sqrt{|mkn(m-k-n)|}} \\
&\times a_0 (a_{-k} a_{n+k-m} + a_{n+k-m} a_{-k}) a_m \\
&-\frac{1}{4A^4}\sum_{m=1(m\neq n)}\sum_{k\neq 0,m}\frac{1}{m^2\sqrt{|n(m-n)k(m-k)|}} a_0 a_{n-m} a_k a_{m-k} \\
&-\frac{1}{12A^4}\sum_{m=1,m\neq n}\sum_{l\neq 0,m}\frac{1}{m^2\sqrt{|ln(m-l)(n-m)|}} a_0 a_l a_{m-l} a_{n-m} \\
&-\frac{1}{12A^4}\sum_{k\neq 0,n}\sum_{l\neq 0,n-k}\frac{1}{(n-k)\sqrt{|knl(n-k-l)|}} \\
&\times a_0\Big(\frac{1}{n-k} a_l a_{n-k-l} a_k + \frac{1}{k} a_k a_l a_{n-k-l}\Big), \quad n > 0.
\end{aligned}
\tag{14}
$$

We only have to pick up terms including $a_0 a_n \hbar^4$ in (7) to calculate $[a_0, a_n]_4$. From (14) and (67) we have

$$
[a_0, a_n]_4 = -\Big(\frac{7}{3n^4} + \frac{5\zeta(2)}{6n^2} + \frac{\zeta(3)}{2n}\Big)\frac{1}{A^4} a_0 a_n. \tag{15}
$$

4 The critical coupling

Using the commutation relations $[a_0, a_n]$ in the previous section, we calculate the vacuum expectation values $\langle 0|[a_0, a_n]_4 a_n^\dagger|0\rangle$ to compute the critical coupling. Sandwiching (9) with $\langle 0|$ and $a_n^\dagger|0\rangle$, we have

$$
\langle 0|[a_0, a_n]_1 a_n^\dagger|0\rangle\hbar = \frac{6}{\tilde{g}}\frac{1}{n}\sigma\hbar^2, \tag{16}
$$

where $\sigma \equiv \langle 0|a_0|0\rangle$ and $\tilde{g} = g + 3\sigma^2$. In the same way from (15) we have

$$\langle 0|[a_0, a_n]_4 a_n^\dagger|0\rangle\hbar^4 = -\Big(\frac{6}{\tilde{g}}\Big)^4\Big(\frac{7}{3n^4} + \frac{5\zeta(2)}{6n^2} + \frac{\zeta(3)}{2n}\Big)\sigma\hbar^5, \tag{17}$$

where $\zeta(n)$ is the Riemann zeta function. Using identities such as $\sum_{k=1}^{n-1}\frac{1}{k(n-k)} = \frac{2}{n}\sum_{k=1}^{n-1}\frac{1}{k}$, from (13) and (14) we have

$$\begin{aligned}\langle 0|[a_0, a_n]_2 a_n^\dagger|0\rangle\hbar^2 \quad &= \quad -\Big(\frac{6}{\tilde{g}}\Big)^2\frac{1}{n^2}\sigma\hbar^3 + \Big(\frac{6}{\tilde{g}}\Big)^3\Big(\frac{1}{4n^3} + \frac{1}{4n^2}\sum_{k=1}^{n-1}\frac{1}{k}\Big)\sigma\hbar^4 \\ &+ \quad \Big(\frac{6}{\tilde{g}}\Big)^4\Big(\frac{1}{2n^2}\sum_{k=1}^{n-1}\frac{1}{k^2} + \frac{1}{2n^3}\sum_{k=1}^{n-1}\frac{1}{k}\Big)\sigma\hbar^5,\end{aligned} \tag{18}$$

$$\begin{aligned}&\langle 0|[a_0, a_n]_3 a_n^\dagger|0\rangle\hbar^3 = \Big(\frac{6}{\tilde{g}}\Big)^3\Big(\frac{4}{3n^3} + \frac{\zeta(2)}{3n}\Big)\sigma\hbar^4 \\ &-\Big(\frac{6}{\tilde{g}}\Big)^4\Big(\frac{1}{3n^4} - \frac{1}{6n^2}\zeta(2) + \frac{17}{6n^3}\sum_{k=1}^{n-1}\frac{1}{k} + \frac{13}{12n^2}\sum_{k=1}^{n-1}\frac{1}{k^2}\Big)\sigma\hbar^5.\end{aligned} \tag{19}$$

The $O(\hbar^4)$ and $O(\hbar^5)$ terms in (18) and $O(\hbar^5)$ terms in (19) originate from the non-diagonal interactions. Collecting the above results, we have

$$\begin{aligned}&\langle 0|[a_0, a_n]a_n^\dagger|0\rangle = \frac{6}{\tilde{g}}\frac{1}{n}\sigma\hbar^2 - \Big(\frac{6}{\tilde{g}}\Big)^2\frac{1}{n^2}\sigma\hbar^3 \\ + \quad &\Big(\frac{6}{\tilde{g}}\Big)^3\Big(\frac{4}{3n^3} + \frac{\zeta(2)}{3n} + \frac{1}{4n^3} + \frac{1}{4n^2}\sum_{k=1}^{n-1}\frac{1}{k}\Big)\sigma\hbar^4 \\ - \quad &\Big(\frac{6}{\tilde{g}}\Big)^4\Big(\frac{7}{3n^4} + \frac{5\zeta(2)}{6n^2} + \frac{\zeta(3)}{2n} + \frac{1}{3n^4} - \frac{\zeta(2)}{6n^2} \\ + \quad &\frac{7}{3n^3}\sum_{k=1}^{n-1}\frac{1}{k} + \frac{7}{12n^2}\sum_{k=1}^{n-1}\frac{1}{k^2}\Big)\sigma\hbar^5.\end{aligned} \tag{20}$$

Sandwiching (4) with the Fock vacuum $|0\rangle$, we have

$$\begin{aligned}0 \quad &= \sigma^3 + g\sigma - \sum_{n=1}^{\infty}\frac{1}{n}\langle 0|[a_0, a_n]a_n^\dagger|0\rangle \\ &= \sigma^3 + g\sigma \\ &-\Big[\frac{6}{\tilde{g}}\zeta(2)\hbar^2 - \Big(\frac{6}{\tilde{g}}\Big)^2\zeta(3)\hbar^3 + \Big(\frac{6}{\tilde{g}}\Big)^3\Big(\frac{19}{12}\zeta(4) + \frac{1}{3}(\zeta(2))^2 + \sum_{n=1}^{\infty}\frac{1}{4n^3}\sum_{k=1}^{n-1}\frac{1}{k}\Big)\hbar^4\end{aligned}$$

$$-\Big(\frac{6}{\tilde{g}}\Big)^4\Big(\frac{8}{3}\zeta(5)+\frac{7}{6}\zeta(3)\zeta(2)+\sum_{n=1}^{\infty}\frac{7}{3n^4}\sum_{k=1}^{n-1}\frac{1}{k}+\sum_{n=1}^{\infty}\frac{7}{12n^3}\sum_{k=1}^{n-1}\frac{1}{k^2}\Big)\hbar^5\Big]\sigma. \tag{21}$$

At the critical point non-trivial solutions start to appear besides the trivial solution $\sigma=0$. From (21) the critical point is given by the condition that the linear terms in σ balance to zero.@This condition is written as

$$0=36-x^2(1-0.2326x+0.1665x^2-0.1068x^3), \tag{22}$$

where $x\equiv 6\pi\hbar/g=\lambda\hbar/4\mu^2$ and we have substituted the values $\sum_{n=1}^{\infty}\frac{1}{n^3}\sum_{k=1}^{n-1}\frac{1}{k}=0.3529$,$\sum_{n=1}^{\infty}\frac{1}{n^3}\sum_{k=1}^{n-1}\frac{1}{k^2}=0.2288$,
$\sum_{n=1}^{\infty}\frac{1}{n^4}\sum_{k=1}^{n-1}\frac{1}{k^2}=0.0966$. Owing to the non-diagonal interactions, the third and the fourth terms in the parentheses in (22) have increased by some 15% and 8% respectively over the previous results in Ref. 9), where the non-diagonal interactions are neglected.

We adopt the Padé approximations[12)] to the alternating series $1-c_1x+c_2x^2-c_3x^3+\cdots$. This series is approximated as $[M/N]$, which denotes a rational equation of polynomials of degrees M and N:

$$[i/1] \qquad 1-c_1x+\cdots+(-1)^ic_i\frac{x^i}{1+\frac{c_{i+1}}{c_i}x}, \tag{23}$$

$$[1/2] \qquad \frac{1+(\frac{c_3-c_1c_2}{c_2-c_1^2}-c_1)x}{1+\frac{c_3-c_1c_2}{c_2-c_1^2}x+(\frac{c_3-c_1c_2}{c_2-c_1^2}c_1-c_2)x^2}. \tag{24}$$

Our results are

$$\begin{array}{llll} [0/0] & \lambda_{\rm cr}=24\mu^2/\hbar, & [1/1] & \lambda_{\rm cr}=28.1\mu^2/\hbar,\\ [0/1] & \lambda_{\rm cr}=46.0\mu^2/\hbar, & [2/1] & \lambda_{\rm cr}=26.0\mu^2/\hbar,\\ [1/2] & \lambda_{\rm cr}=25.5\mu^2/\hbar. & & \end{array} \tag{25}$$

In Ref. 9) we have assumed that the alternating series has such properties that the lower(upper) bounds of $\lambda_{\rm cr}$ are obtained from cases $M+N$=even(odd). The values in (25) exhibit that this assumption does not hold. However, all the values in (25) still lie within the range $22\mu^2/\hbar\le\lambda_{\rm cr}\le 55.5\mu^2/\hbar$ of the equal-time result.

The ratio of the bare coupling constant to the bare mass is the only parameter of the theory and in the literature $g_{\rm cr}$ is computed. In two dimension the coupling constant and the mass suffer only finite quantum corrections in this model. However, in four dimension the bare quantities become infinite and it will be adequate to describe the theory by renormalized quantities. Hence as a preliminary to study four-dimensional models, we adopt the ratio of the bare coupling constant to a renormalized mass as a new parameter in the next

section. We avoid treating coupling constant renormalization since it seems not so easy to calculate quantum corrections to sufficient order in $\hbar$. In the next section we compute the critical value of the bare coupling constant in terms of the renormalized mass.

5 The critical coupling in terms of the renormalized mass

Let us study a relation between the bare mass μ and the renormalized mass m for the model. We split the Hamiltonian into two parts $H = \frac{d\mu^2}{4\pi}(H_0 + H_I)$, where $H_0 = g\sum_2 = g\sum_{n=1}^{\infty}\frac{a_n^\dagger a_n}{n}$ and $H_I = 6\sum_4 + H_{ZM}$. The Hamiltonian H_{ZM} contains all of the zero-mode interactions

$$\begin{aligned} H_{ZM} &= -\frac{a_0^4}{4} + \frac{1}{4}\sum_{n\neq 0}\frac{1}{|n|}(a_n a_0^2 a_{-n} - a_0 a_n a_{-n} a_0) \\ &+ \frac{1}{4}\sum_{k,l,n\neq 0}\frac{\delta_{k+l+n,0}}{\sqrt{|kln|}}(a_k a_0 a_l a_n + a_k a_l a_0 a_n). \end{aligned} \tag{26}$$

The states $a_p^\dagger|0\rangle$ are eigenstates of the unperturbed Hamiltonian H_0 with eigenvalues $\frac{g}{p}\hbar = \frac{24\pi\mu^2}{p\lambda}\hbar$.@We approximately define the renormalized mass m through eigenvalues of H as

$$Ha_p^\dagger|0\rangle \approx \frac{24\pi m^2}{p\lambda}\hbar a_p^\dagger|0\rangle. \tag{27}$$

If the interactions were absent we would obtain the bare mass μ, hence this definition of the renormalized mass m will be plausible. We will solve the eigenvalue equation (27) perturbatively.

Effects of the interaction H_I have already been calculated to the second order in perturbation theory in the limit $d \to \infty$ in Ref. 8). Taking (22) into consideration, we calculate effects of the interaction H_I to the third order in perturbation theory in the limit $d \to \infty$.

First, we calculate the contributions of the non-zero modes. Since we have added the counter term in (4), the tad pole diagrams always vanish. At the second order in perturbation theory we have a non-vanishing diagram (see Fig. 1):

$$\begin{aligned} &\langle 0|a_p(6\textstyle\sum_4)\dfrac{1}{E - g\sum_2}(6\textstyle\sum_4)a_p^\dagger|0\rangle \\ = \;& \frac{6}{gp}\sum_{i,j>0}^{i+j<p}\frac{1}{ip(p-i-j)\left(\frac{1}{p}-\frac{1}{i}-\frac{1}{j}-\frac{1}{p-i-j}\right)}\hbar^4 \\ \longrightarrow \;& -\frac{6}{gp}\int_0^P dx\int_0^{P-x}dy\frac{P}{(x+y)(P-x)(P-y)}\hbar^4 = -\frac{6}{gp}\frac{\pi^2}{4}\hbar^4, \end{aligned} \tag{28}$$

where i, j and p are natural numbers and $E = \frac{g}{p}\hbar$ is the unperturbed energy; we have set $x = 2\pi i/d, y = 2\pi j/d$ and $P = 2\pi p/d$ and have taken the limit $d, p \to \infty$ so that P remains finite. Equation (22) is independent of d and it holds also at large d. Therefore in the following we consider the situation $d \to \infty$. In the ordinary equal-time formulation the corresponding Feynman diagram of Fig. 1 gives the same result.[8)] In the same way at the third order in perturbation theory(see Fig. 2) we have

$$\langle 0|a_p(6\sum{}_4)\frac{1}{E - g\sum_2}(6\sum{}_4)\frac{1}{E - g\sum_2}(6\sum{}_4)a_p^\dagger|0\rangle = \frac{54}{pg^2}I\hbar^5, \tag{29}$$

where I is the following quantity which approaches a definite value in the limit $d \to \infty$

$$\begin{aligned} I = & \\ & \sum_{i,j,k>0}^{k<i+j<p} \frac{1}{ip(p-i-j)\left(\frac{g}{p} - \frac{1}{i} - \frac{1}{j} - \frac{1}{p-i-j}\right)k(i+j-k)\left(\frac{g}{p} - \frac{1}{k} - \frac{1}{i+j-k} - \frac{1}{p-i-j}\right)} \\ \longrightarrow & \\ & \int_0^P dx \int_0^{P-x} dy \int_0^{x+y} dz \frac{P^2(P-x-y)}{(x+y)^2(P-x)(P-y)(P-x-y+z)} = 1.645. \end{aligned} \tag{30}$$

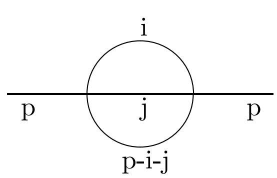

Fig. 1. The second order diagram.

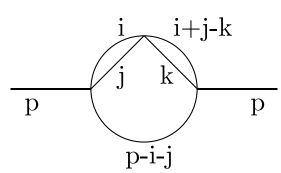

Fig. 2. The third order diagram.

Second, we calculate the contributions of H_{zm}. Solving the constraint equation (4) perturbatively in $1/g$ for a_0, we have

$$a_0 = -\frac{6}{g}\sum{}_3 + \frac{6}{g^2}\sum_{n\neq 0}\left(\sum{}_3 a_n a_{-n} + a_n a_{-n}\sum{}_3 + a_n\sum{}_3 a_{-n} - \frac{3}{2}\sum{}_3\right) + \cdots. \tag{31}$$

Substituting this expression into H_{zm} (26), to $O(\hbar^5)$ we find

$$\langle 0|a_p H_{zm} a_p^\dagger|0\rangle = -\frac{3}{gp}\Big(\sum_{k=1}^{p-1}\frac{1}{k(p-k)} + 4\sum_{k>0}\frac{1}{k(p+k)}\Big)\hbar^4$$
$$+\frac{18}{g^2 p}\Big(\sum_{k=1}^{p-1}\frac{1}{k^2(p-k)} + \frac{4}{p}\sum_{k=1}^{p-1}\frac{1}{k(p-k)}$$
$$+\sum_{k>0}\frac{1}{k^2(p+k)} + \sum_{k>0}\frac{1}{k(p+k)^2} - \frac{1}{p}\sum_{k>0}\frac{1}{k(p+k)}\Big)\hbar^5, \qquad (32)$$

where we have neglected some constant terms, which means a shift of the vacuum energy. Note that $\sum_{k=1}^{p-1}\frac{1}{k(p-k)}$ and $\sum_{k>0}\frac{1}{k(p+k)}$ in the $O(\hbar^4)$ terms vanish in the limit $p\to\infty$. The $O(\hbar^5)$ terms also vanish in this limit. Thus the mass will not be corrected by the zero mode in this limit. However, the convergences in (28) and (30) will be improved by inclusions of the zero mode.[8)]

The eigenvalues of H_0 and their corrections (28) and (29) have the common factor $1/p$ as expected, and we get the following relation between the renormalized mass m and the bare mass μ

$$\begin{aligned} m^2 &= \mu^2\big(1 - \frac{6}{g^2}\frac{\pi^2}{4}\hbar^2 + \frac{54}{g^3}I\hbar^3\big) + O(\hbar^4) \\ &= \mu^2\big(1 - \frac{14.80}{g^2}\hbar^2 + \frac{88.83}{g^3}\hbar^3\big) + O(\hbar^4). \end{aligned} \qquad (33)$$

Now we redefine the parameter g as $g = 24\pi m^2/\lambda$, which we regard as independent of $\hbar$, instead of $g = 24\pi\mu^2/\lambda$. In the following g means this new parameter. From (33) we have

$$\frac{24\pi\mu^2}{\lambda} = g + \frac{14.80}{g}\hbar^2 - \frac{88.83}{g^2}\hbar^3 + O(\hbar^4). \qquad (34)$$

Taking this $\hbar$ dependence into account, the constraint equation (7) becomes

$$\text{l.h.s. of (7)} + \frac{14.80}{6g}[a_0,a_n]\hbar^2 - \frac{88.83}{6g^2}[a_0,a_n]\hbar^3 = 0. \qquad (35)$$

These two terms affect $[a_0,a_n]_2$ and $[a_0,a_n]_3$

$$[a_0,a_n]_3 = \text{r.h.s. of (7)} - \frac{14.80}{6gA^2}\frac{1}{n}a_0 a_n, \qquad (36)$$

$$[a_0,a_n]_4 = \text{r.h.s. of (7)} + \frac{14.80}{3gA^3}\frac{1}{n^2}a_0 a_n + \frac{88.83}{6g^2A^2}\frac{1}{n}a_0 a_n. \qquad (37)$$

Thus the fluctuations in the $\hbar$ expansion of $[a_0,a_n]$ have diminished in terms of the renormalized mass. This time we have instead of (22)

$$0 = 36 - x^2(1 - 0.23261x + 0.124889x^2 - 0.0742x^3), \qquad (38)$$

with $x = \frac{\lambda\hbar}{4m^2}$.

Using again the Padé approximations, this time we have

$$\begin{array}{llll} [0/0] & \lambda_{\rm cr} = 24m^2/\hbar, & [1/1] & \lambda_{\rm cr} = 29.7m^2/\hbar, \\ [0/1] & \lambda_{\rm cr} = 46.0m^2/\hbar, & [2/1] & \lambda_{\rm cr} = 33.3m^2/\hbar, \\ [1/2] & \lambda_{\rm cr} = 32.2m^2/\hbar. & & \end{array} \tag{39}$$

The variance of the values in (39) is smaller than that of the values in (25). The values in (39) seem to approach their mean value as the order $M+N$ of the Padé approximations increases unlike the case (25).

6 Summary

Taking the non-diagonal interactions into account besides the diagonal interactions, we have calculated the relevant terms of $[a_0, a_n]$ to $O(\hbar^4)$. In addition to the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, the more complicated function $\sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{k=1}^{n-1} \frac{1}{k^t}$ has appeared corresponding to the non-diagonal interactions. Owing to the non-diagonal interactions, the third and fourth coefficients of the alternating series in (22) have increased by some 15% and 8%. We have applied the Padé approximations to the alternating series to compute the critical coupling. Although the critical coupling shifts sensitively to the coefficients in the $\hbar$ expansion, at each order of the Padé approximations $\lambda_{\rm cr}$ lies within the range $22\mu^2/\hbar \le \lambda_{\rm cr} \le 55.5\mu^2/\hbar$ of the equal-time result. We also have calculated the renormalized mass in front-form perturbation theory. We have obtained another $\hbar$ expansion of $[a_0, a_n]$ by regarding the ratio of the bare coupling constant to the renormalized mass as a new independent parameter. We have found that the fluctuations in the $\hbar$ expansion of $[a_0, a_n]$ and the variance of the values of the critical coupling have diminished.

Acknowledgments

The author would like to thank Dr.M.Yahiro, who invited him to study this subject.

Appendix A@ $[\frac{1}{A}, a_n]$

In this appendix we calculate the fundamental commutation relations $[\frac{1}{A}, a_n]$. We can obtain $[\frac{1}{A}, a_m]$ from $[a_0, a_m]$ as follows

$$\left[\frac{1}{A}, a_m\right] = \frac{1}{A}[a_m, A]\frac{1}{A}$$

$$
\begin{aligned}
&= \frac{1}{A}\Big([a_m, \frac{1}{2}a_0^2] + \frac{\hbar}{m}a_m\Big)\frac{1}{A} \\
&= \Big(\frac{\hbar}{m}\frac{1}{A^2} - \Big(\frac{\hbar}{m}\Big)^2\frac{1}{A^3} + \Big(\frac{\hbar}{m}\Big)^3\frac{1}{A^4} + \cdots\Big)a_m - \frac{1}{mA^3}a_0^2 a_m \hbar \\
&- \frac{1}{2A^3}\sum_{k\neq 0,m}\frac{1}{\sqrt{|km(m-k)|}}a_k a_{m-k}a_0\hbar + O(\hbar^2), m > 0.
\end{aligned}
\tag{40}
$$

The abbreviated terms lead to higher-order contributions. In the same way we have

$$
\begin{aligned}
\Big[\frac{1}{A}, a_m^\dagger\Big] &= -\Big(\frac{\hbar}{m}\frac{1}{A^2} + \Big(\frac{\hbar}{m}\Big)^2\frac{1}{A^3} + \Big(\frac{\hbar}{m}\Big)^3\frac{1}{A^4} + \cdots\Big)a_m^\dagger + \frac{1}{mA^3}a_0^2 a_m^\dagger \hbar \\
&+ \frac{1}{2A^3}\sum_{k\neq 0,m}\frac{1}{\sqrt{|km(m-k)|}}a_{m-k}^\dagger a_k^\dagger a_0\hbar + O(\hbar^2), m > 0.
\end{aligned}
\tag{41}
$$

Equations (40) and (41) are combined as

$$
\begin{aligned}
\Big[\frac{1}{A}, a_n\Big] &= \Big(\frac{\hbar}{n}\frac{1}{A^2} - \Big(\frac{\hbar}{n}\Big)^2\frac{1}{A^3} + \Big(\frac{\hbar}{n}\Big)^3\frac{1}{A^4} + \cdots\Big)a_n - \frac{1}{nA^3}a_0^2 a_n \hbar \\
&- \frac{\epsilon(n)}{2A^3}\sum_{k\neq 0,n}\frac{1}{\sqrt{|kn(n-k)|}}a_k a_{n-k}a_0\hbar + O(\hbar^2), n \neq 0.
\end{aligned}
\tag{42}
$$

Appendix B $[a_0, a_m^\dagger a_m]$

In this appendix we calculate the first few terms of $[a_0, a_m^\dagger a_m]$ in the $\hbar$ expansion. In our approximation it is sufficient to get $[a_0, a_m^\dagger a_m]_p (p = 1, 2, 3)$.@ More precisely, we need the coefficients of the terms in the expansion

$$
[a_0, a_m^\dagger a_m] = (a_0 aa\hbar + a_0 aa\hbar^2 + a_0 aaaa\hbar^2 + a_0 aa\hbar^3) + (aaa\hbar), \tag{43}
$$

where the parentheses denote the disregard of the coefficients and a's represent the non-zero modes. We have neglected terms such as $(a_0^3 aa\hbar^2)$ and $(aaa\hbar^2)$ that become higher order in $\hbar$.

Let us examine how $[a_0, a_m^\dagger a_m]$ would be if the non-diagonal interactions were absent. At the lowest level we have

$$
\begin{aligned}
[a_0, a_m^\dagger a_m]_1 &= a_m^\dagger [a_0, a_m]_1 + [a_0, a_m^\dagger]_1 a_m \\
&\sim a_m^\dagger \frac{1}{A}a_0 a_m \hbar - a_m^\dagger a_0 \frac{1}{A}a_m\hbar = a_m^\dagger \Big[\frac{1}{A}, a_0\Big]a_m\hbar \\
&\sim \sum_l a_m^\dagger [a_l^\dagger a_l, a_0]a_m\hbar.
\end{aligned}
\tag{44}
$$

This equation will indicate that if we neglect the non-diagonal interactions, $[a_0, a_m^\dagger a_m]$ can vanish at any order of $\hbar$. In the following we calculate the coefficients of (43) taking the non-diagonal interactions into account.

First, we calculate the contributions of the diagonal parts $[a_0, a_m]_{\mathrm{d}p}$ $(p = 1, 2, 3)$. Using (11) and (13), we have

$$\begin{aligned}[a_0, a_m^\dagger]_{\mathrm{d1}} a_m \hbar \quad &+ \quad a_m^\dagger [a_0, a_m]_{\mathrm{d1}} \hbar = \frac{2}{A^2 m^2} a_0 a_m^\dagger a_m \hbar^2 + \frac{3}{A^3 m^3} a_0 a_m^\dagger a_m \hbar^3 @ \\ &\oplus \quad (a_0^3 aa\hbar^2 + aaa\hbar^2 + a_0^2 aaa\hbar^2 + a_0 aaaa\hbar^3). \end{aligned} \tag{45}$$

Here the symbol $\oplus$ means that the following terms can be neglected in our approximation. In the following we usually neglect such higher-order terms that become $O(\hbar^6)$ in $\langle 0|[a_0, a_n]a_n^\dagger|0\rangle$. The $O(\hbar)$ terms have canceled, since $-\frac{1}{m} + \frac{1}{m} = 0$. In the same way we have

$$\begin{aligned} & [a_0, a_m^\dagger]_{\mathrm{d2}} a_m \hbar^2 + a_m^\dagger [a_0, a_m]_{\mathrm{d2}} \hbar^2 \\ = \quad & -\frac{2}{A^2 m^2} a_0 a_m^\dagger a_m \hbar^2 - \frac{3}{A^3 m^3} a_0 a_m^\dagger a_m \hbar^3 \\ + \quad & \frac{1}{8A^3} \sum_{k \neq 0, m} \sum_{l \neq 0, n-k} \frac{1}{(m-k)\sqrt{|mkl(m-l-k)|}} a_0 (a_m^\dagger a_l a_{m-k-l} a_k \\ + \quad & a_m^\dagger a_k a_l a_{m-k-l} + a_l^\dagger a_{m-k-l}^\dagger a_k^\dagger a_m + a_k^\dagger a_l^\dagger a_{m-k-l}^\dagger a_m) \hbar^2. \end{aligned} \tag{46}$$

The last term derives from the first ϵ^2-term in (13). At $O(\hbar^3)$ we only have to pay attention to the terms including $a_0 a_m^\dagger a_m \hbar^3$. Since $-\frac{1}{m} + \frac{1}{m} = -\frac{1}{m^3} + \frac{1}{m^3} = 0$, from the first term of (14) we have

$$[a_0, a_m^\dagger]_{\mathrm{d3}} a_m \hbar^3 + a_m^\dagger [a_0, a_m]_{\mathrm{d3}} \hbar^3 = 0\hbar^3 \oplus (a_0 aaaa\hbar^3). \tag{47}$$

Second, we calculate the contributions of the non-diagonal parts $[a_0, a_m^\dagger a_m]_{\mathrm{nd}p}(p = 1, 2)$. At the lowest level we have

$$\begin{aligned} & [a_0, a_m^\dagger]_{\mathrm{nd1}} a_m \hbar + a_m^\dagger [a_0, a_m]_{\mathrm{nd1}} \hbar \\ = \quad & a_m^\dagger \frac{1}{2A} \sum_{k \neq 0, m} \frac{1}{\sqrt{|km(m-k)|}} a_k a_{m-k} \hbar \\ - \quad & \frac{1}{2A} \sum_{k \neq 0, m} \frac{1}{\sqrt{|km(m-k)|}} a_{-k} a_{-m+k} a_m \hbar \\ = \quad & \frac{\epsilon(m)}{2A} \sum_{k \neq 0, m} \frac{1}{\sqrt{|km(m-k)|}} (a_m^\dagger a_k a_{m-k} - a_{-k} a_{k-m} a_m) \hbar \\ - \quad & \frac{\epsilon^2(m)}{4A^3} \sum_{l \neq 0, m} \sum_{k \neq 0, m} \frac{1}{m\sqrt{|l(m-l)k(m-k)|}} a_0 a_{-l} a_{l-m} a_k a_{m-k} \hbar^2 \\ \oplus \quad & \frac{1}{2}\Big(\frac{1}{mA^2} + \frac{\hbar}{m^2 A^3} + \cdots\Big) \sum_{k \neq 0, m} \frac{\epsilon(m)}{\sqrt{|km(m-k)|}} a_m^\dagger a_k a_{m-k} \hbar^2. \end{aligned} \tag{48}$$

Since $[\frac{1}{A}aaa\hbar^p, a_n] = (aa\hbar^{p+1}) + (a_0aaaaa\hbar^{p+1})$, the contributions of the first and third terms in (48) to $\langle 0|[a_0, a_n]a_n^\dagger|0\rangle$ are $O(\hbar^5)$ and $O(\hbar^6)$ respectively. The second term in (48) gives non-zero contributions to $[a_0, a_m a_m^\dagger]_{\mathrm{d3}}$. Hence we cannot neglect the second term. In the same way we have

$$[a_0, a_m^\dagger]_{\mathrm{nd2}} a_m \hbar^2 + a_m^\dagger [a_0, a_m]_{\mathrm{nd2}} \hbar^2 = \oplus(aaa\hbar^2). \tag{49}$$

The right hand side of (49) gives only higher-order contributions. Thus as concerned with the non-diagonal interactions, we only have to take the lowest-order terms into account .

Collecting the above results for the diagonal and the non-diagonal interactions, we obtain the main result in this appendix:

$$\begin{aligned}
& [a_0, a_m^\dagger a_m] \\
= \; & \frac{1}{2A} \sum_{k\neq 0,m} \frac{1}{\sqrt{|km(m-k)|}} (a_m^\dagger a_k a_{m-k} - a_m a_k^\dagger a_{m-k}^\dagger)\hbar \\
+ \; & \frac{1}{8A^3} \sum_{k\neq 0,m} \sum_{l\neq 0,n-k} \frac{1}{(m-k)\sqrt{|mkl(m-l-k)|}} \\
\times \; & a_0 (a_m^\dagger a_l a_{m-k-l} a_k + a_m^\dagger a_k a_l a_{m-k-l} \\
& + a_l^\dagger a_{m-k-l}^\dagger a_k^\dagger a_m + a_k^\dagger a_l^\dagger a_{m-k-l}^\dagger a_m)\hbar^2 \\
- \; & \frac{1}{4A^3} \sum_{k\neq 0,m} \sum_{l\neq 0,m} \frac{1}{m\sqrt{|lk(m-l)(m-k)|}} a_0 a_{-l} a_{-m+l} a_k a_{m-k} \hbar^2 .@
\end{aligned} \tag{50}$$

Using (50), we can write down the lowest term of the expansion of $[[a_0, a_m a_m^\dagger], a_n]$ as

$$\begin{aligned}
& [[a_0, a_m a_m^\dagger]_1, a_n]_1 \hbar^2 \\
= \; & -\frac{\epsilon(n)}{4A^3} \sum_{l\neq 0,n} \sum_{k\neq 0,m} \frac{\epsilon(m)}{\sqrt{|ln(n-l)km(m-k)|}} \\
\times \; & a_0 a_{n-l} a_l (a_m^\dagger a_k a_{m-k} - a_m a_k^\dagger a_{m-k}^\dagger)\hbar^2 \\
+ \; & \frac{\epsilon(n)}{2} \sum_{k\neq 0,n} \Big(-\frac{1}{n} + \frac{1}{k} + \frac{1}{n-k} \Big) \frac{1}{A^2} \frac{1}{\sqrt{|nk(n-k)|}} a_k a_{n-k} \hbar^2 \\
+ \; & \frac{1}{2n} \Big(\frac{1}{A^3} - \frac{1}{A^4} a_0^2 \Big) \sum_{k\neq 0,m} \frac{\epsilon(m)}{\sqrt{|mk(m-k)|}} a_n (a_m^\dagger a_k a_{m-k} - a_m a_k^\dagger a_{m-k}^\dagger)\hbar^2, \\
& n \neq 0.
\end{aligned} \tag{51}$$

The first term becomes $O(\hbar^5)$ when sandwiched between $\langle 0|$ and $a_n^\dagger|0\rangle$. The second term leads to $O(\hbar^5)$ contributions through $a_0[[a_0, a_n], a_0]$(see (60)). The

next higher-order term $[[a_0, a_m a_m^\dagger]_1, a_n]_2\hbar^3$ produces only higher-order terms. The commutation relations $[[a_0, a_m^\dagger a_m]_2, a_n]_1$ contribute to $[a_0, a_n]_3$ in the form $(a_0 aaa\hbar^3)$:

$$
\begin{aligned}
& \sum_{m=1} \frac{1}{m}[[a_0, a_m a_m^\dagger]_2, a_n]_1\hbar^3 \\
= \quad & -\frac{1}{8A^3}\sum_{k\neq 0,n}\sum_{l\neq 0,n-k}\frac{1}{n(n-k)\sqrt{|nkl(n-k-l)|}} \\
& \times a_0(a_l a_{n-k-l} a_k + a_k a_l a_{n-k-l})\hbar^3 \\
- \quad & \frac{1}{4A^3}\sum_{m=1}\sum_{k\neq 0,m,m-n}\frac{1}{m(m-k)\sqrt{|mkn(m-k-n)|}} \\
& \times a_0(a_{m-n-k}^\dagger a_k^\dagger + a_k^\dagger a_{m-n-k}^\dagger)a_m\hbar^3 \\
+ \quad & \frac{1}{2A^3}\sum_{m=1(m\neq n)}\sum_{k\neq 0,m}\frac{1}{m^2\sqrt{|n(m-n)k(m-k)|}}a_0 a_{n-m} a_k a_{m-k}\hbar^3 \\
\ominus \quad & \frac{1}{4A^3}\sum_{m=1(m\neq n)}\sum_{k\neq 0,m-n}\frac{1}{m(m-n)\sqrt{|mkn(m-k-n)|}}a_0 a_k^\dagger a_{m-n-k}^\dagger a_m\hbar^3 \\
\oplus \quad & (a_0 a_m^\dagger aa + a_0 aaa_{m+n} + \cdots)\hbar^3.
\end{aligned}
\tag{52}
$$

The terms in the last two lines become higher order, since $a_m a_n^\dagger|0\rangle = 0 (m \neq n)$ and $\langle 0|a_m^\dagger = a_{m+n}a_n^\dagger|0\rangle = 0$, etc..

Appendix C $\quad a_0[[a_0, a_n], a_0]$

In this appendix we calculate $a_0[[a_0, a_n], a_0]$. Expanding $[a_0, a_n]$ in $\hbar$, we have

$$
a_0[[a_0, a_n], a_0] = a_0[[a_0, a_n]_1, a_0]\hbar + a_0[[a_0, a_n]_2, a_0]\hbar^2. \tag{53}
$$

We calculate these terms by dividing $[a_0, a_n]_p (p = 1, 2)$ into the diagonal and non-diagonal parts.

First, we calculate the first term in (53). The contribution of the diagonal part of $[a_0, a_n]_1$ is calculated as

$$
\begin{aligned}
& a_0[[a_0, a_n]_{\mathrm{d}1}, a_0]\hbar = a_0\left[\frac{1}{nA}a_0 a_n, a_0\right]\hbar \\
= \quad & a_0\frac{1}{n}\left[\frac{1}{A}, a_0\right]a_0 a_n\hbar - a_0\frac{1}{nA}a_0[a_0, a_n]\hbar \\
= \quad & \oplus(a_0 aaaa_0 a\hbar^2) \oplus \left(a_0\frac{1}{A}a_0^2 a\hbar^2\right) \oplus @\left(a_0\frac{1}{A}a_0 aa\hbar^2\right) \\
= \quad & \ominus(a_0 aaaa_0 a\hbar^2) \ominus \frac{1}{n^2A^2}a_0^3 a_n\hbar^2 \\
& \ominus\frac{1}{2nA^2}\sum_{k\neq 0,n}\frac{\epsilon(n)}{\sqrt{|kn(n-k)|}}a_0^2 a_k a_{n-k}\hbar^2,
\end{aligned}
\tag{54}
$$

where we have used $[\frac{1}{A}, a_0] \sim aaa\hbar$. The three terms in the last line only produce higher-order terms. Thus within our approximation the contribution of the diagonal part of $[a_0, a_n]$ is negligible.@The contribution of the non-diagonal part of $[a_0, a_n]_1$ is calculated as

$$
\begin{aligned}
& a_0[[a_0, a_n]_{\mathrm{nd1}}, a_0]\hbar \\
= \quad & a_0 \sum_{k\neq 0,n} \frac{\epsilon(n)}{2\sqrt{|kn(n-k)|}} \Big[\frac{1}{A} a_k a_{n-k}, a_0\Big]\hbar \\
= \quad & a_0 \sum_{k\neq 0,n} \frac{\epsilon(n)}{2\sqrt{|kn(n-k)|}} \Big(\Big[\frac{1}{A}, a_0\Big] a_k a_{n-k} - \frac{1}{A} a_k [a_0, a_{n-k}] \\
& -\frac{1}{A}[a_0, a_k] a_{n-k}\Big)\hbar. \qquad (55)
\end{aligned}
$$

The first term in the parentheses in (55) is calculated as

$$
\begin{aligned}
& a_0 \sum_{k\neq 0,n} \frac{\epsilon(n)}{2\sqrt{|kn(n-k)|}} \Big[\frac{1}{A}, a_0\Big] a_k a_{n-k}\hbar \\
= \quad & \sum_{k\neq 0,n} \frac{\epsilon(n)}{4A^2\sqrt{|kn(n-k)|}} \sum_{m=1} \sum_{l\neq 0,m} \frac{1}{m\sqrt{|lm(m-l)|}} a_0 (a_m^\dagger a_l a_{m-l} \\
& -a_m a_l^\dagger a_{m-l}^\dagger) a_k a_{n-k} \hbar^2 @ \oplus (a_0 aaaaa\hbar^3). \qquad (56)
\end{aligned}
$$

At the lowest level the second term in the parentheses in (55) is calculated as

$$
\begin{aligned}
- \quad & a_0 \frac{1}{2A} \sum_{k\neq 0,n} \frac{\epsilon(n)}{\sqrt{|kn(n-k)|}} a_k [a_0, a_{n-k}]_1 \hbar^2 \\
= \quad & -a_0 \frac{1}{2A^2} \sum_{k\neq 0,n} \frac{\epsilon(n)}{(n-k)\sqrt{|kn(n-k)|}} a_k \frac{1}{A} a_0 a_{n-k} \hbar^2 \\
& -a_0 \frac{1}{2A} \sum_{k\neq 0,n} \frac{\epsilon(n)}{\sqrt{|kn(n-k)|}} @ \\
& \times a_k \sum_{l\neq 0,n-k} \frac{1}{2A} \frac{\epsilon(n-k)}{\sqrt{|l(n-k)(n-k-l)|}} a_l a_{n-k-l} \hbar^2 \\
= \quad & -\frac{1}{2A^2} \sum_{k\neq 0,n} \frac{\epsilon(n)}{(n-k)\sqrt{|kn(n-k)|}} a_0^2 a_k a_{n-k} \hbar^2 @ \\
- \quad & \frac{1}{4A^2} \sum_{k\neq 0,n} \sum_{l\neq 0,n-k} \frac{\epsilon(n)}{(n-k)\sqrt{|knl(n-k-l)|}} a_0 a_k a_l a_{n-k-l} \hbar^2 @ \\
+ \quad & \frac{1}{4A^3} \sum_{k\neq 0,n} \sum_{l\neq 0,n-k} \frac{\epsilon(n)}{(n-k)k\sqrt{|knl(n-k-l)|}} \\
& \times a_0 (a_k a_l a_{n-k-l} + a_l a_{n-k-l} a_k) \hbar^3, \qquad (57)
\end{aligned}
$$

where we have used $\epsilon(k)/|k| = 1/k$ and changes of variables $k \to n-k$. In (57) we have obtained the two $O(\hbar^3)$ terms by commuting a_k with a_0 and $1/A$. At the next level the second term in (55) is calculated from $[a_0, a_{n-k}]_2$ as

$$\begin{aligned}
- & \quad a_0 \frac{1}{2A} \sum_{k\neq 0,n} \frac{\epsilon(n)}{\sqrt{|kn(n-k)|}} a_k [a_0, a_{n-k}]_2 \hbar^3 \\
= & \quad \frac{1}{8A^3} \sum_{k\neq 0,n} \sum_{l\neq 0,n-k} \frac{\epsilon(n)}{(n-k)\sqrt{|knl(n-k-l)|}} \Big(\frac{1}{l} + \frac{1}{n-k-l} + \frac{1}{n-k} \Big) \\
& \quad \times a_0 a_k a_l a_{n-k-l} \hbar^3. \qquad (58)
\end{aligned}$$

The third term in the parentheses in (55) is calculated more easily as

$$\begin{aligned}
- & \quad a_0 \frac{1}{2A} \sum_{k\neq 0,n} \frac{\epsilon(n)}{\sqrt{|kn(n-k)|}} ([a_0, a_k]_1 a_{n-k} \hbar^2 + [a_0, a_k]_2 a_{n-k} \hbar^3) \\
= & \quad -\frac{1}{2A} \sum_{k\neq 0,n} \frac{\epsilon(n)}{k\sqrt{|kn(n-k)|}} a_0^2 a_k a_{n-k} \hbar^2 \\
@ \quad - & \quad \frac{1}{4A^2} \sum_{k\neq 0,n} \sum_{l\neq 0,n-k} \frac{\epsilon(n)}{(n-k)\sqrt{|knl(n-k-l)|}} a_0 a_l a_{n-k-l} a_k \hbar^2 \\
+ & \quad \frac{1}{8A^3} \sum_{k\neq 0,n} \sum_{l\neq 0,n-k} \frac{\epsilon(n)}{(n-k)\sqrt{|knl(n-k-l)|}} \Big(\frac{1}{l} + \frac{1}{n-k-l} + \frac{1}{n-k} \Big) \\
& \quad \times a_0 a_l a_{n-k-l} a_k \hbar^3, \qquad (59)
\end{aligned}$$

where we have used changes of variables $k \to n-k$.

Second, we calculate the second term in (53). As for the lowest term $a_0[[a_0, a_n]_2, a_0]_1 \hbar^3$, only the non-diagonal interactions give non-zero contributions

$$\begin{aligned}
& \quad a_0 [[a_0, a_n]_{\mathrm{nd}2}, a_0]_{\mathrm{nd}1} \hbar^3 \\
= & \quad a_0 \Big[a_0, \frac{\epsilon(n)}{4A^2} \sum_{k\neq 0,n} \Big(\frac{1}{k} + \frac{1}{n-k} + \frac{1}{n} \Big) \frac{1}{\sqrt{|nk(n-k)|}} a_k a_{n-k} \Big]_{\mathrm{nd}1} \hbar^3 \\
= & \quad \frac{\epsilon(n)}{8A^3} \sum_{k\neq 0,n} \sum_{l\neq 0,k} \frac{1}{(n-k)\sqrt{|nkl(n-k-l)|}} \Big(\frac{1}{n} + \frac{1}{k} + \frac{1}{n-k} \Big) \\
\times & \quad a_0 (a_k a_l a_{n-k-l} + a_l a_{n-k-l} a_k) \hbar^3. \qquad (60)
\end{aligned}$$

Other terms in the third term in (53) give higher-order terms.

Collecting (56)$\sim$(60), we have

$$\begin{aligned}
& \quad a_0 [[a_0, a_n], a_0] \\
= & \quad \sum_{k\neq 0,n} \sum_{m=1} \sum_{l\neq 0,m} \frac{\epsilon(n)}{4A^2 m \sqrt{|kn(n-k)lm(m-l)|}}
\end{aligned}$$

$$
\begin{aligned}
\times \quad & a_0(a_m^\dagger a_l a_{m-l} - a_m a_l^\dagger a_{m-l}^\dagger) a_k a_{n-k} \hbar^2 \\
- \quad & \frac{1}{4A^2} \sum_{k\neq 0,n} \sum_{l\neq 0,n-k} \frac{\epsilon(n)}{(n-k)\sqrt{|knl(n-k-l)|}} \\
& \times a_0(a_k a_l a_{n-k-l} + a_l a_{n-k-l} a_k)\hbar^2 @ \\
- \quad & \frac{\epsilon(n)}{2A^2} \sum_{k\neq 0,n} \Big(\frac{1}{n} + \frac{1}{k} + \frac{1}{n-k}\Big) \frac{1}{\sqrt{|kn(n-k)|}} a_0^2 a_k a_{n-k} \hbar^2 @ \\
+ \quad & \frac{\epsilon(n)}{8A^3} \sum_{k\neq 0,n} \sum_{l\neq 0,n-k} \frac{1}{(n-k)\sqrt{|nkl(n-k-l)|}} \Big(\frac{1}{n} + \frac{3}{k} + \frac{2}{n-k} + \frac{1}{l} \\
& + \frac{1}{n-k-l}\Big) a_0(a_k a_l a_{n-k-l} + a_l a_{n-k-l} a_k)\hbar^3. \qquad (61)
\end{aligned}
$$

Appendix D $\quad [[[a_0, a_n], a_0], a_0]$

The lowest-order term in $[[[a_0, a_n], a_0], a_0]$ is $[[[a_0, a_n]_1, a_0]_1, a_0]_1 \hbar^3$. In these commutation relations each $[\frac{1}{A}, a_0]_1$ produces only higher-order terms. There are only three ways to get relevant terms. Choosing one diagonal interaction and two non-diagonal interactions for the three brackets, we have

$$
\begin{aligned}
& [[[a_0, a_n]_1, a_0]_1, a_0]_1 \hbar^3 \\
= \quad & \frac{1}{4A^3} \sum_{k\neq 0,n} \sum_{l\neq 0,n-k} \frac{\epsilon(n)\epsilon(n-k)}{|n-k|\sqrt{|nkl(n-k-l)|}} \\
\times \quad & \Big((\frac{1}{n} + \frac{1}{l} + \frac{1}{n-k-l} + \frac{1}{k}) a_0(a_l a_{n-k-l} a_k + a_k a_l a_{n-k-l}) \\
+ \quad & (\frac{2}{n-k} a_0 a_l a_{n-k-l} a_k + \frac{2}{k} a_0 a_k a_l a_{n-k-l}) \Big) \hbar^3. \qquad (62)
\end{aligned}
$$

The other terms in $[[[a_0, a_n], a_0], a_0]$ produce only higher-order terms.

Appendix E $\quad [[[a_0, a_n], a_m], a_m^\dagger]$

First, we calculate the lowest-order term in $[[[a_0, a_n], a_m], a_m^\dagger]$:

$$
\begin{aligned}
[[[a_0, a_n]_1, a_m]_1, a_m^\dagger]_1 \hbar^3 &= \Big[\Big[\frac{1}{nA} a_0 a_n, a_m\Big]_1, a_m^\dagger\Big]_1 \hbar^3 \\
&+ \sum_{k\neq 0,n} \frac{\epsilon(n)}{\sqrt{|kn(n-k)|}} \Big[\Big[\frac{1}{2A} a_k a_{n-k}, a_m\Big]_1, a_m^\dagger\Big]_1 \hbar^3. \qquad (63)
\end{aligned}
$$

In the first term in (63) the non-diagonal interactions give higher-order terms. From the diagonal interactions we have

$$
\Big[\Big[\frac{1}{nA} a_0 a_n, a_m\Big]_1, a_m^\dagger\Big]_1 \hbar^3
$$

$$
\begin{aligned}
= \quad & \frac{2}{nmA^3}a_0a_n\hbar^3 + \delta_{n,m}\frac{1}{n^2A^2}(a_na_0 + a_0a_n)\hbar^3 - \frac{1}{nm^2A^3}a_0a_ma_m^\dagger a_n\hbar^3 \\
& \oplus(aa + a_0aaa_{n+m})\hbar^3 \\
= \quad & \frac{2}{nmA^3}a_0a_n\hbar^3 + \delta_{n,m}\frac{2}{n^2A^2}a_0a_n\hbar^3 - \delta_{n,m}\frac{1}{n^3A^3}a_0a_n\hbar^4 \\
& -\frac{1}{nm^2A^3}a_0a_n\hbar^4 \oplus (aa + a_0aaa_{n+m} + a_0a_m^\dagger aa)\hbar^3, n > 0. \qquad (64)
\end{aligned}
$$

As for the second term in (63) only the non-diagonal interactions in $[\frac{1}{A}, a_m]$ produce non-zero contributions

$$
\begin{aligned}
& \sum_{k\neq 0,n} \frac{\epsilon(n)}{\sqrt{|kn(n-k)|}}\Big[\Big[\frac{1}{2A}a_ka_{n-k}, a_m\Big]_1, a_m^\dagger\Big]_1\hbar^3 \\
= \quad & -\frac{1}{2A^3}\sum_{l\neq 0,m}(1-\delta_{n,m})\frac{\epsilon(n)\epsilon(m)}{m\sqrt{|ln(m-l)(n-m)|}}a_0a_la_{m-l}a_{n-m}\hbar^3, \\
& n > 0. \qquad (65)
\end{aligned}
$$

Second, we calculate the next-order term $[[[a_0, a_n]_i, a_m]_j, a_m^\dagger]_k\hbar^4 \quad (i+j+k = 4)$. In these commutation relations only the diagonal interactions produce non-zero contributions. Paying attention to the commutation relations $[\frac{1}{A}, a_m]$, we have

$$
\begin{aligned}
& [[[a_0, a_n]_2, a_m]_1, a_m^\dagger]_1 + [[[a_0, a_n]_1, a_m]_2, a_m^\dagger]_1 \\
= \quad & -\Big[\Big[\frac{1}{n^2A^2}a_0a_n, a_m\Big]_1, a_m^\dagger\Big]_1 + \Big[\Big[\frac{1}{nA}a_0a_n, a_m\Big]_2, a_m^\dagger\Big]_1 \\
= \quad & -\Big(\frac{3}{n^2m} + \frac{2}{nm^2}\Big)(1+\delta_{n,m})\frac{1}{A^3}a_0a_m\hbar^4, n > 0. \qquad (66)
\end{aligned}
$$

The rest of $O(\hbar^4)$ terms $[[[a_0, a_n]_1, a_m]_1, a_m^\dagger]_2\hbar^4$ produces only higher-order contributions like $(a_0aaa\hbar^4)$. From (64),(65) and (66) we have

$$
\begin{aligned}
& \frac{1}{6A}\sum_{m=1}\frac{1}{m}[[[a_0, a_n], a_m], a_m^\dagger] \\
= \quad & \frac{\zeta(2)}{3nA^3}a_0a_n\hbar^3 + \frac{1}{3n^3A^3}a_0a_n\hbar^3 \\
- \quad & \sum_{m=1,m\neq n}\sum_{l\neq 0,m}\frac{1}{12m^2\sqrt{|ln(m-l)(n-m)|}A^4}a_0a_la_{m-l}a_{n-m}\hbar^3 \\
- \quad & \frac{1}{n^4A^4}a_0a_n\hbar^4 - \frac{\zeta(2)}{2n^2A^4}a_0a_n\hbar^4 - \frac{\zeta(3)}{2nA^4}a_0a_n\hbar^4, n > 0. \qquad (67)
\end{aligned}
$$

References

1)For a review, see S. J. Brodsky, H-C. Pauli and S. S. Pinsky, Phys. Rep. **301**

(1998), 299.
2)S. J. Chang, R. G. Root and T. M. Yan, Phys. Rev. D7,1973,1133.
S. J. Chang and T. M. Yan, Phys. Rev. D7,1973,1147.
3)T. Maskawa and K. Yamawaki, Prog. Thor. Phys. 56,1976,270.
4)S. J. Chang, Phys. Rev. D13,1976,2778.
J. Abad, J. G. Esteve and A. F. Pacheco, Phys. Rev. D32,1985,2729.
M. Funke, U. Kaulfuss and H. Kummel, Phys. Rev. D35,1987,621.
H. Kroger, R. Girard and G. Dufour, Phys. Rev. D35,1987,3944.
5)T. Heinzl, S. Krusche, S. Simburger and E. Werner, Z. Phys.**C56**(1992), 415.
T. Heinzl, S. Krusche and E. Werner, Phys. Lett. B272,1991,54.
6)D. G. Robertson, Phys. Rev. D47,1993,2549.
7)C. M. Bender, S. S. Pinsky and B. van de Sande, Phys. Rev. D48,1993,816.
S. S. Pinsky and B. van de Sande, Phys. Rev. D49,1994,2001.
8)S. Pinsky, B. van de Sande and J. R. Hiller, Phys. Rev. D51,1995,726. 9)K. Oshima and M. Yahiro, Prog. Thor. Phys. 101,1999,459.
10) To discretize the momentum space is advantageous also in calculating the hadron spectra; see, H. -C. Pauli and S .J. Brodsky, D32,1985,1993, 2001.
11) C. M. Bender, L .R. Mead and S. S. Pinsky, Phys. Rev. Lett. 56,1986,2445.
12)J. Killingbeck, *Microcomputer Quantum Mechanics*(Adam Hilger, Bristol, 1983).